Retake test 2 Numerical Mathematics 2
February, 2022

Duration: One hour.

In front of the questions one finds the points. The sum of the points plus 1 gives the end mark of this test. Use of a calculator is allowed.

1. Consider the 100×100 matrix

$$A = \begin{bmatrix} -1 & 101 & & & & \\ -99 & -2 & 101 & & & \\ & -99 & -2 & 101 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -99 & -2 & 101 \\ & & & & -99 & -2 \end{bmatrix}.$$

   (a) [0.5] Show that A is irreducible.

   (b) [1.5] Write A as the sum of a symmetric and a skew-symmetric matrix. Consider the symmetric part and localize its eigenvalues by the Gershgorin theorems. And similar for the skew-symmetric part of A

   (c) [0.5] According to Bendixson's theorem where are the eigenvalues of A located in the complex plane based on the results in the previous part?

2. (a) [1.0] Describe the QR-method when applied to a matrix A. To what type of matrix does the sequence $\{A^{(n)}\}$ converge? What can be said about the ordering of the eigenvalues in the result and the speed of convergence?

   (b) [1.0] To speedup the convergence of the QR-method one uses a shift. How is the shift chosen in general and why does it improve the convergence?

   (c) [2.0] Determine the Householder matrix that transforms the vector $[1, -\sqrt{3}]^T$ in a vector of the shape $[*, 0]^T$ where $*$ is a nonzero number. It is enough to find the vector v that defines the Householder matrix and indicate how it defines the matrix. Explain the advantages of using v instead of H when computing Hx for general x (look at both storage and computing effort).

   (d) [1.5] Explain how the Householder matrix derived in the previous part can be used to bring the matrix

$$A = \begin{bmatrix} -1 & 1 & \sqrt{5} \\ -1 & -2 & 6 \\ \sqrt{3} & -4 & -7 \end{bmatrix}$$

   in Hessenberg form using a similarity transformation.

   (e) [1.0] For general matrices, the transformation of a matrix to Hessenberg form, as in part b, is used as a preprocessing step for the QR-method. Explain why applying the QR method to a matrix of Hessenberg form is advantageous compared to applying it to a general (dense) matrix A.